Linear Integer Arithmetic Revisited

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July 17, 2015

Linear Integer Arithmetic

Input:

$$C = \bigwedge_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \leq b_i \right), \text{ where } a_{ij}, b_i \in \mathbb{Z}$$

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Example: $-2 \leq 1 \quad \wedge \quad 0 \geq 0 \quad \wedge \quad 2 \geq 2$

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Difficulty

Restricting solutions to Integers vs Reals:

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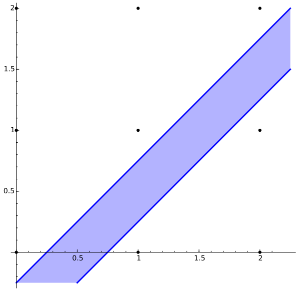
$$\Rightarrow \bigvee_{k=0}^{2} 3 \mid z + k \quad \wedge \quad 3y \geq 4z + 4k \quad \wedge \quad z + k \geq z$$

Unbounded Problems

$$4x - 4y \geq 1$$
$$\wedge$$
$$4x - 4y \leq 3$$



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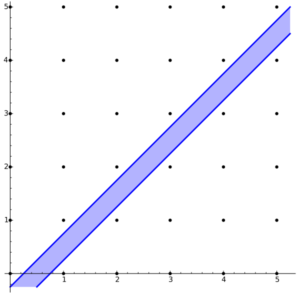

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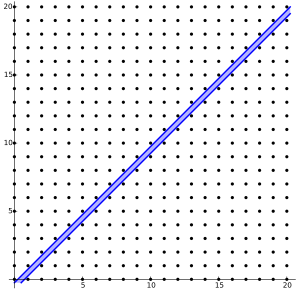

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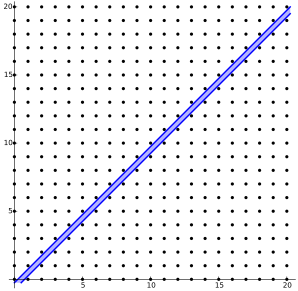

A Priori Bounds:

- $\forall i. \; -n(ma)^{2m+1} \leq x_i \leq n(ma)^{2m+1}$
- finite search space: solve via enumeration
- for the example: $-65536 \leq x, y \leq 65536$
- in practice: too large!
- but: state-of-the-art!

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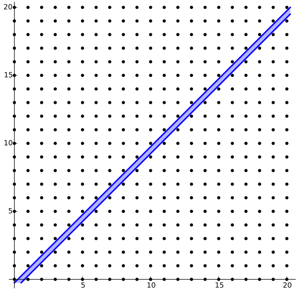

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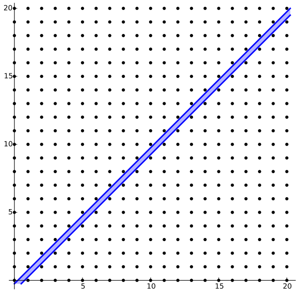

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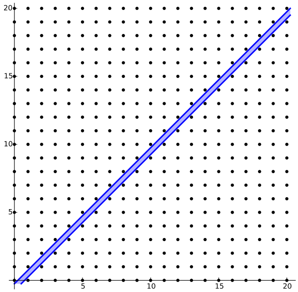

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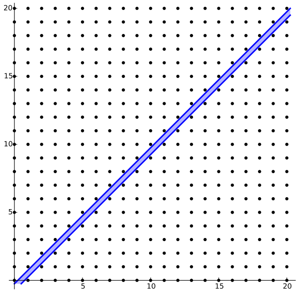

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by Dejan Jovanovic and Leonardo M. de Moura (2011, 2013)

A CDCL based LIA solver (CUTSAT)

- termination oriented on problem structure
- termination is dynamic; not by a priori bounds
- termination based on quantifier elimination
  - reducing unbounded variables to bounded variables
  - requires divisibility constraints, e.g., $6 \mid 3x + 2y$

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$$C := x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ z \geq 0 \wedge \quad z \geq y + 1 \quad \wedge \quad z \leq x + y$$

$$M := [\![]\!]$$

Propagate

Propagate inequalities with one variable!

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Use model to propagate more bounds!

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Find responsible constraints

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Done

Complete model: $x := 1, y := 0, z := 1$

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Claims:

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$$C := x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ y \leq 1 \ \wedge \quad \begin{matrix} 2 \mid 2 \cdot 0 + 1 \\ 2 \mid 2x + y \end{matrix}$$

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Conflict

Find responsible constraints under Assignment $x = 0$, $y = 1$;

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$$2 \mid 2 \cdot 0 + 1$$

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Conflict

CUTSAT is stuck! But not at a solution, e.g., $x = 0$, $y = 0$

Finding Gaps

Find and fill gaps by turning:

dynamic termination strategy

$\Downarrow$

static quantifier elimination

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Result: New QE procedure (Weak Cooper Elimination)

Weak Cooper Elimination

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Output:	- a conjunction C' without x; - $\exists K.C' \equiv \exists x.C(x)$
Drawback:	introduce new but bounded variables
Termination:	eliminate all unbounded variables

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Divide constraints into cores:

If: an assignment ν is not a solution for $C(x)$

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3.Diophantine:	0	0		

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2.Divisibility:	1	1	1	$2 \geq x \quad \wedge \quad x \geq 1 \quad \wedge \quad 3 \mid x$
3.Diophantine:	0	0	1	$2 \mid 2x + 1$

Cores:

Divide constraints into cores:

If: an assignment ν is not a solution for $C(x)$

Then: there exists a responsible core that is conflicting under ν

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Note: Cores are not necessarily conflicting, e.g., $y \geq x \quad \wedge \quad x \geq z$

Weak Cooper Elimination

$$\exists x.C(x) \equiv \exists k_1, \ldots, k_{n'}. \left[\left(\bigwedge_{\text{all } I \text{ in } C(x) \text{ w/o } x} I \right) \wedge \left(\bigwedge_{\text{for each core } C^*} R_{C^*} \right) \right]$$

Resolvents R_{C^*}:

- R_{C^*} free of x
- R_{C^*} describes C^*'s solutions for x
- $k_1, \ldots, k_{n'}$ are new bounded variables

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Example 1

Input:

$$3x \geq y \quad \wedge \quad x \leq 4z$$

After eliminating x:

$$\exists k.(k \geq 0 \wedge k \leq 2 \quad \wedge \quad 3 \mid y + k \quad \wedge \quad y + k \leq 12z)$$

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$$x := \frac{y+k}{3} \in \mathbb{Z} \qquad \frac{y}{3} \leq \frac{y+k}{3} \leq 4z$$

Example 2

Input:

$$3x \geq y \quad \wedge \quad 2x \geq 2y - z \quad \wedge \quad x \leq 4z$$

After eliminating x:

$$\exists k_1.(k_1 \geq 0 \wedge k_1 \leq 2 \quad \wedge \quad 3 \mid y + k_1 \quad \wedge \quad y + k_1 \leq 12z)$$

$$\bigwedge$$

$$\exists k_2.(k_2 \geq 0 \wedge k_2 \leq 1 \quad \wedge \quad 2 \mid 2y - z + k_2 \quad \wedge \quad 2y - z + k_2 \leq 8z)$$

Example 3

Input:

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Solution for x from resolvent:

as y is even any $x \geq 0$

CUTSAT++

Idea:

- describe conflicts as cores
- learn resolvents to prevent a core from reverting into a conflict

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- CUTSAT: only first two types of cores $\Rightarrow$ Stuck
- CUTSAT++: all types of cores

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$$C := x \geq 0 \;\wedge\; y \geq 0 \;\wedge\; y \leq 1 \;\wedge\; \begin{array}{l} 2 \mid 2 \cdot 0 + 1 \\ 2 \mid 2x + y \end{array}$$

$$M := [\![x \geq 0, y \geq 0, y \leq 1, y \geq_{\mathbf{d}} 1, x \leq_{\mathbf{d}} 0]\!]$$

Conflict

CUTSAT finds no core! CUTSAT is stuck!

Example

$$2 \mid 2 \cdot 0 + 1$$

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Conflict

But CUTSAT++ finds a diophantine core!

Example

$$C := x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ y \leq 1 \wedge \quad \begin{matrix} 2 \mid 2 \cdot 0 + 1 \\ 2 \mid 2x + y \end{matrix}$$

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Conflict

Backtrack and block conflict with resolvent!

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Propagate

Propagate divisibility constraint!

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Decision

Assign variables with a decision!

Example

$$C := x \geq 0 \;\wedge\; y \geq 0 \;\wedge\; y \leq 1 \wedge \quad 2 \mid 2x + y \quad \wedge 2 \mid y$$

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Decision

Assign variables with a decision!

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$$M := [\![x \geq 0, y \geq 0, y \leq 1, y \leq 0, x \leq_{\mathbf{d}} 0]\!]$$

Done

Complete model: $x := 0, y := 0$

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CUTSAT++:

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