NP-completeness of small conflict set generation for congruence closure

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Sorry for giving a 25 minute presentation,
but I did not have time to prepare a 5 minute one.

Conflict Set

- Unsatisfiable set of equations and negated equations

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Example

- $\{g(c_1, \ldots, c_n) = d,\ f(a) = a,\ a = b,\ b = f(b),\ f(a) \neq f(b)\}$

Conflict Set

- Unsatisfiable set of equations and negated equations

Example

- $\{ \quad f(a) = a, \, a = b, \, b = f(b), \, f(a) \neq f(b)\}$

Conflict Set

- Unsatisfiable set of equations and negated equations

Example

- $\{ \quad f(a) = a,\, a = b,\, b = f(b),\, f(a) \neq f(b)\}$
- Transitivity

Conflict Set

- Unsatisfiable set of equations and negated equations

Example

- $\{a = b, b = f(b), f(a) \neq f(b)\}$
- Transitivity

Conflict Set

- Unsatisfiable set of equations and negated equations

Example

- $\{ \quad a = b, b = f(b), f(a) \neq f(b)\}$
- Transitivity
- Congruence: $t_1 = s_1$ and $\ldots$ $t_n = s_n$ implies
$$f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n)$$

Conflict Set

- Unsatisfiable set of equations and negated equations

Example

- $\{ \quad a = b, \quad f(a) \neq f(b)\}$
- Transitivity
- Congruence: $t_1 = s_1$ and $\ldots$ $t_n = s_n$ implies
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Why do we want small conflict sets? (1)

- Speed up SMT decision procedures

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$\boxed{\text{Input SMT problem } \Psi}$

Why do we want small conflict sets? (1)

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$$\begin{tikzcd} \boxed{\text{Input SMT problem } \Psi} \arrow[d, "\text{Treat equations as propositional variables}"] \\ \boxed{\text{Propositional logic abstraction } \phi} \end{tikzcd}$$

Why do we want small conflict sets? (1)

- Speed up SMT decision procedures

Input SMT problem Ψ

$\downarrow$ Treat equations as propositional variables

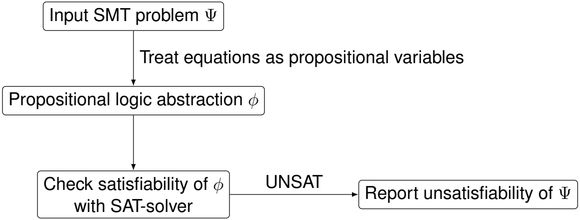
Propositional logic abstraction ϕ

$\downarrow$

Check satisfiability of ϕ with SAT-solver

Why do we want small conflict sets? (1)

- Speed up SMT decision procedures



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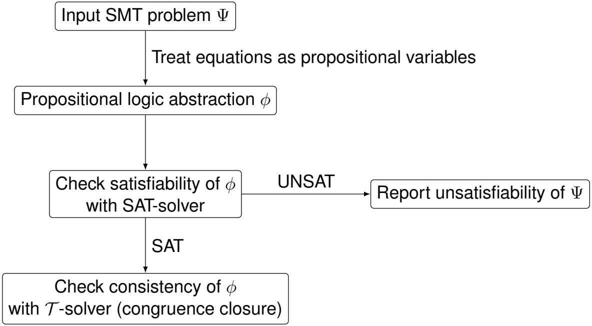

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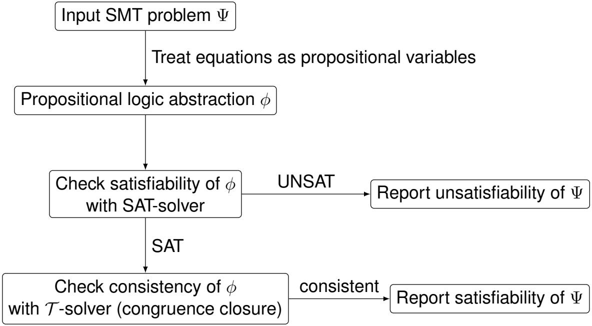

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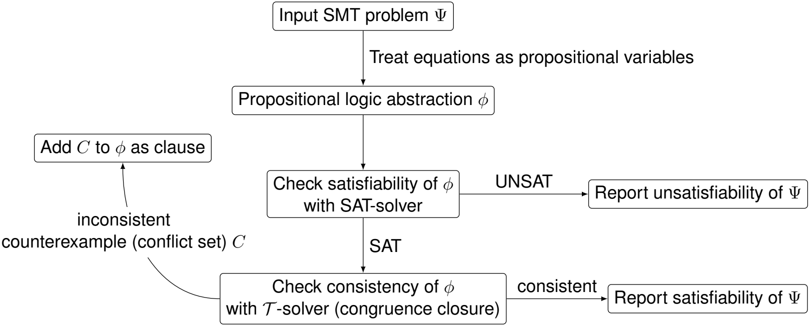

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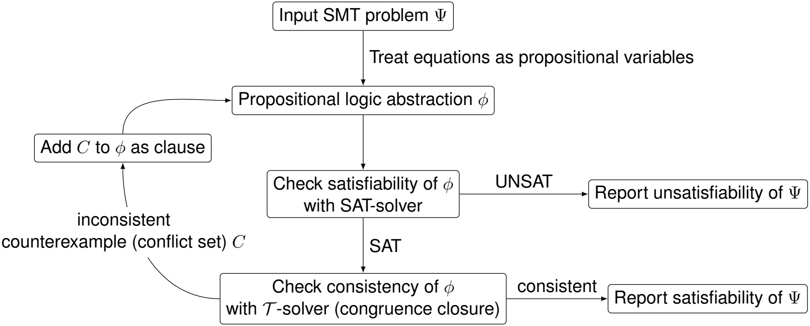
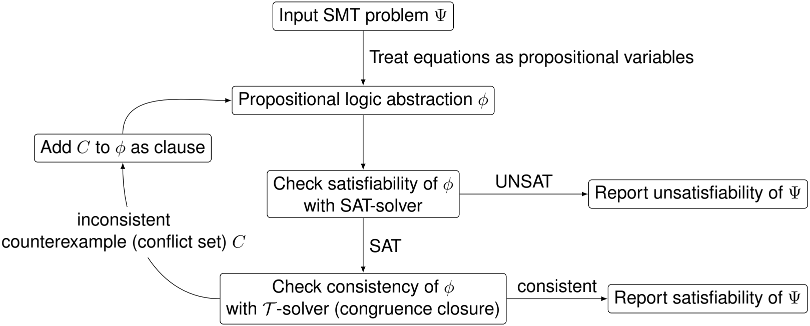


Why do we want small conflict sets? (1)

- Speed up SMT decision procedures



Smaller conflict set

- Eliminate more spurious counterexamples at once
- Fewer loops

Why do we want small conflict sets? (2)

- Smaller proofs
- Proof corresponding to transitivity

$$
\begin{tikzcd}
f(a) \neq a, a \neq b, b \neq f(b), f(a) = f(b) & & & f(a) = a & \\
& a \neq b, b \neq f(b), f(a) = f(b) \arrow[ul] \arrow[urr] & & a = b & \\
& & b \neq f(b), f(a) = f(b) \arrow[ul] \arrow[ur] & & b = f(b) \\
& f(a) \neq f(b) & & f(a) = f(b) \arrow[ul] \arrow[ur] & \\
& & \perp \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Why do we want small conflict sets? (2)

- Smaller proofs
- Proof corresponding to congruence

$$
\begin{tikzcd}
& a \neq b, f(a) = f(b) & & a = b \\
f(a) \neq f(b) & & f(a) = f(b) \arrow[ul] \arrow[ur] & \\
& \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

Conflict Set vs Explanation

Explanation for $s = t$

- Set of equations E, such that $E \models s = t$
- $E \cup \{s \neq t\}$ is a conflict set

Conflict set C

- There is $s \neq t \in C$, such that
- $C \setminus \{s \neq t\}$ is an explanation for $s = t$

Small Explanation Decision Problem

Given a set of equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

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NP-complete

Small Explanation Decision Problem

Given a set of equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

NP-complete

Small Explanation is in NP

1. Guess $E' \subseteq E$, which is polynomial in input size.
2. Check $E' \models s = t$ with congruence closure algorithm in polynomial time.

NP-hardness

Reduction of SAT

- Given a propositional logic formula in CNF
  $\phi = C_1 \wedge \cdots \wedge C_n$
- Using variables $x_1, \ldots, x_m$
- Construct a set of equations E and a target equation $s = t$, such that

ϕ is satisfiable

if and only if

There exists an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq 3n+4m-1$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Example of Reduction

$$\phi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

c_1 c_2 c_3

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

c_1 $\quad c_1' - c_2$ $\quad c_2' - c_3$ $\quad c_3'$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

c_1 $\quad c_1' - c_2$ $\quad c_2' - c_3$ $\quad c_3'$

$\hat{x}_1$

$\hat{x}_2$

$\hat{x}_3$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

$c_1 \qquad c_1' - c_2 \qquad c_2' - c_3 \qquad c_3'$

$\bot_1$ —— $\hat{x}_1$ —— $\top_1$

$\bot_2$ —— $\hat{x}_2$ —— $\top_2$

$\bot_3$ —— $\hat{x}_3$ —— $\top_3$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

$$\begin{array}{llll} & t_1(\hat{x}_1) & & \\ & / & & \\ c_1 & - \; t_1(\hat{x}_2) & c_1' - c_2 \qquad c_2' - c_3 & \qquad c_3' \\ & \backslash & & \\ & t_1(\hat{x}_3) & & \end{array}$$

$$\bot_1 \text{ —— } \hat{x}_1 \text{ —— } \top_1$$

$$\bot_2 \text{ —— } \hat{x}_2 \text{ —— } \top_2$$

$$\bot_3 \text{ —— } \hat{x}_3 \text{ —— } \top_3$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

$$\begin{tikzcd}
& t_1(\hat{x}_1) & t_1(\top_1) \arrow[dr, dash] & & & & \\
c_1 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & t_1(\hat{x}_2) & t_1(\top_2) \arrow[r, dash] & c_1' \arrow[r, dash] & c_2 & c_2' \arrow[r, dash] & c_3 & c_3' \\
& t_1(\hat{x}_3) & t_1(\bot_3) \arrow[ur, dash] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\bot_1 \arrow[r, dash] & \hat{x}_1 \arrow[r, dash] & \top_1 \\
\bot_2 \arrow[r, dash] & \hat{x}_2 \arrow[r, dash] & \top_2 \\
\bot_3 \arrow[r, dash] & \hat{x}_3 \arrow[r, dash] & \top_3
\end{tikzcd}$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

$$\begin{array}{ccccccc}
 & t_1(\hat{x}_1) & t_1(\top_1) & & & t_2(\hat{x}_2) & \\
 \diagup & & & \diagdown & & \diagup & \\
c_1 & - \, t_1(\hat{x}_2) & t_1(\top_2) \, - & c'_1 & - \; c_2 & & c'_2 - c_3 \qquad c'_3 \\
 \diagdown & & & \diagup & & \diagdown & \\
 & t_1(\hat{x}_3) & t_1(\bot_3) & & & t_2(\hat{x}_3) &
\end{array}$$

$$\bot_1 \;\text{——}\; \hat{x}_1 \;\text{——}\; \top_1$$

$$\bot_2 \;\text{——}\; \hat{x}_2 \;\text{——}\; \top_2$$

$$\bot_3 \;\text{——}\; \hat{x}_3 \;\text{——}\; \top_3$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

$$t_1(\hat{x}_1) \quad t_1(\top_1)$$

$$c_1 - t_1(\hat{x}_2) \quad t_1(\top_2) - c'_1 - c_2 \qquad t_2(\hat{x}_2) \quad t_2(\bot_2) \qquad c'_2 - c_3 \qquad\qquad c'_3$$

$$t_1(\hat{x}_3) \quad t_1(\bot_3) \qquad t_2(\hat{x}_3) \quad t_2(\top_3)$$

$$\bot_1 \text{ --- } \hat{x}_1 \text{ --- } \top_1$$

$$\bot_2 \text{ --- } \hat{x}_2 \text{ --- } \top_2$$

$$\bot_3 \text{ --- } \hat{x}_3 \text{ --- } \top_3$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

$$\begin{tikzcd}
 & t_1(\hat{x}_1) & t_1(\top_1) \arrow[ddr, dash] & & & & & & & & \\
 & & & & & t_2(\hat{x}_2) & t_2(\bot_2) \arrow[dr, dash] & & & t_3(\hat{x}_1) & \\
c_1 \arrow[uur, dash] \arrow[r, dash] \arrow[ddr, dash] & t_1(\hat{x}_2) & t_1(\top_2) \arrow[r, dash] & c'_1 \arrow[r, dash] & c_2 \arrow[ur, dash] \arrow[dr, dash] & & & c'_2 \arrow[r, dash] & c_3 \arrow[ur, dash] \arrow[dr, dash] & & c'_3 \\
 & & & & & t_2(\hat{x}_3) & t_2(\top_3) \arrow[ur, dash] & & & t_3(\hat{x}_3) & \\
 & t_1(\hat{x}_3) & t_1(\bot_3) \arrow[uur, dash] & & & & & & & &
\end{tikzcd}$$

$$\bot_1 \;\text{——}\; \hat{x}_1 \;\text{——}\; \top_1$$

$$\bot_2 \;\text{——}\; \hat{x}_2 \;\text{——}\; \top_2$$

$$\bot_3 \;\text{——}\; \hat{x}_3 \;\text{——}\; \top_3$$

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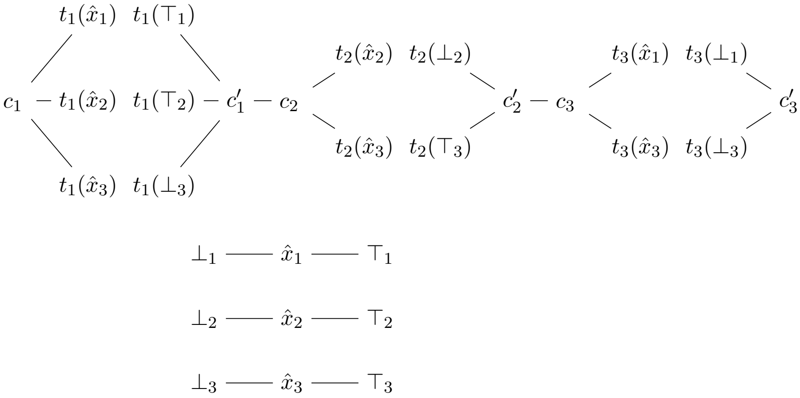
Equations E $(a - b \rightsquigarrow a = b \in E)$

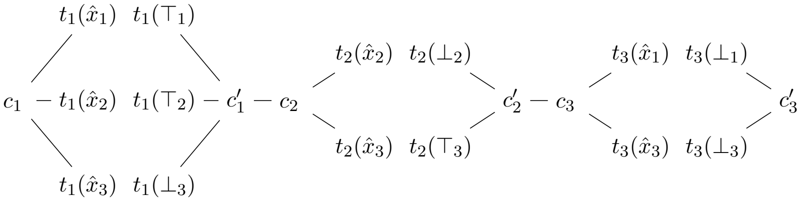


Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$$\bot_1 \text{ —— } \hat{x}_1 \text{ —— } \top_1$$

$$\bot_2 \text{ —— } \hat{x}_2 \text{ —— } \top_2$$

$$\bot_3 \text{ —— } \hat{x}_3 \text{ —— } \top_3$$

Proof Arguments

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- Translate assignment $\mathcal{I}$ to subset of equations E':

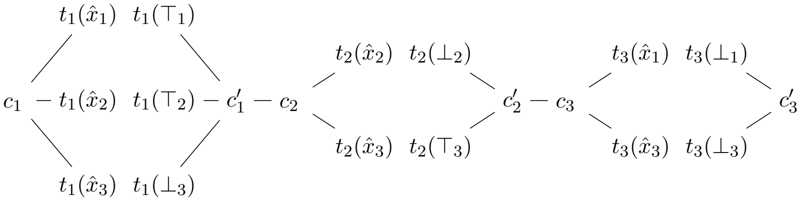
$$x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \top_i \in E'$$
$$\neg x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \bot_i \in E'$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E ($a - b \rightsquigarrow a = b \in E$)

Target Equation $c_1 = c'_3$



$$\bot_1 \text{ ------ } \hat{x}_1 \text{ ------ } \top_1$$

$$\bot_2 \text{ ------ } \hat{x}_2 \text{ ------ } \top_2$$

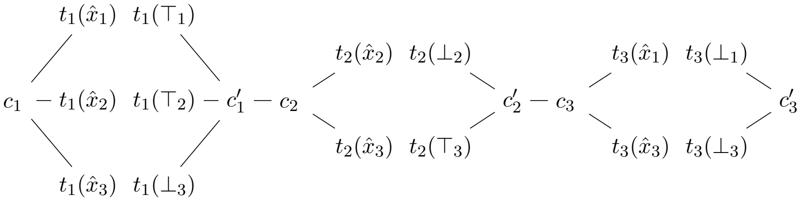
$$\bot_3 \text{ ------ } \hat{x}_3 \text{ ------ } \top_3$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$\perp_1$ —— $\hat{x}_1$ —— $\top_1$

$\perp_2$ —— $\hat{x}_2$ —— $\top_2$

$\perp_3$ —— $\hat{x}_3$ —— $\top_3$

Assignment:

Example of Reduction

$$\phi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$

$$\begin{tikzcd}[column sep=small]
 & t_1(\hat{x}_1) & t_1(\top_1) \arrow[dr, dash] & & & & & & & & & & \\
 & & & & & & t_2(\hat{x}_2) & t_2(\bot_2) \arrow[dr, dash] & & & t_3(\hat{x}_1) & t_3(\bot_1) \arrow[dr, dash] & \\
c_1 \arrow[uur, dash] \arrow[r, dash] \arrow[ddr, dash] & t_1(\hat{x}_2) & t_1(\top_2) \arrow[r, dash] & c_1' \arrow[r, dash] & c_2 \arrow[urr, dash] \arrow[drr, dash] & & & & c_2' \arrow[r, dash] & c_3 \arrow[ur, dash] \arrow[dr, dash] & & & c_3' \\
 & & & & & & t_2(\hat{x}_3) & t_2(\top_3) \arrow[ur, dash] & & & t_3(\hat{x}_3) & t_3(\bot_3) \arrow[ur, dash] & \\
 & t_1(\hat{x}_3) & t_1(\bot_3) \arrow[uur, dash] & & & & & & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\bot_1 \arrow[r, dash] & \hat{x}_1 \arrow[r, dash] & \top_1 \\
\bot_2 \arrow[r, dash] & \hat{x}_2 \arrow[r, dash] & \top_2 \\
\bot_3 \arrow[r, dash] & \hat{x}_3 \arrow[r, dash] & \top_3
\end{tikzcd}$$

Assignment:

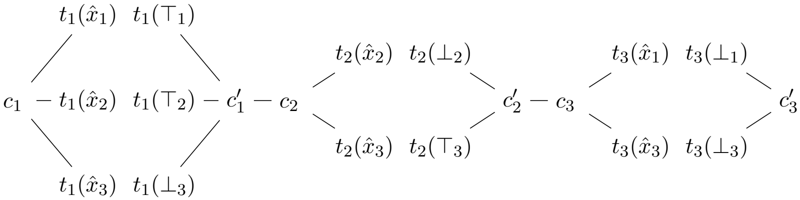
$$\mathcal{I}_1 = \{x_1, \neg x_2, \neg x_3\}$$

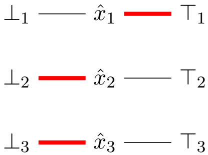
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$





Assignment:

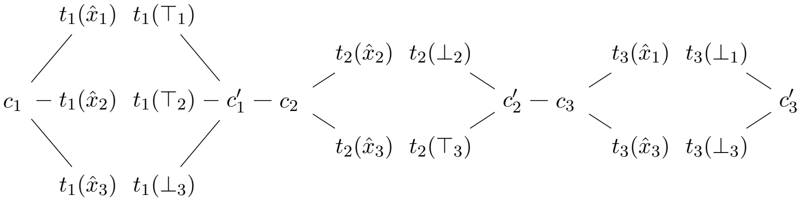
$$\mathcal{I}_1 = \{x_1, \neg x_2, \neg x_3\}$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c'_3$



$\bot_1$ —— $\hat{x}_1$ —— $\top_1$

$\bot_2$ —— $\hat{x}_2$ —— $\top_2$

$\bot_3$ —— $\hat{x}_3$ —— $\top_3$

Assignment:

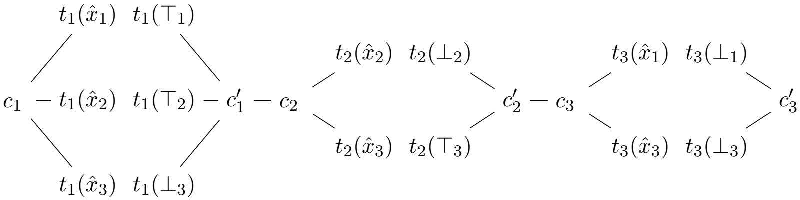
$\mathcal{I}_2 = \{\neg x_1, \neg x_2, x_3\}$

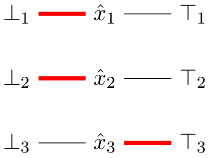
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E ($a - b \rightsquigarrow a = b \in E$)

Target Equation $c_1 = c_3'$





Assignment:

$$\mathcal{I}_2 = \{\neg x_1, \neg x_2, x_3\}$$

Proof Arguments

- Translate assignment $\mathcal{I}$ to subset of equations E':

$$\begin{aligned} x_i \in \mathcal{I} &\Leftrightarrow \hat{x}_i = \top_i \in E' \\ \neg x_i \in \mathcal{I} &\Leftrightarrow \hat{x}_i = \bot_i \in E' \end{aligned}$$

Proof Arguments

- Translate assignment $\mathcal{I}$ to subset of equations E':

$$x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \top_i \in E'$$
$$\neg x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \bot_i \in E'$$

- Every short explanation contains the translation of an assignment

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$

$$\begin{tikzcd}[column sep=small]
 & t_1(\hat{x}_1)\ \ t_1(\top_1) \arrow[dr, no head] & & & t_2(\hat{x}_2)\ \ t_2(\bot_2) \arrow[dr, no head] & & & t_3(\hat{x}_1)\ \ t_3(\bot_1) \arrow[dr, no head] & \\
c_1 \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] & t_1(\hat{x}_2)\ \ t_1(\top_2) \arrow[r, no head] & c_1' \arrow[r, no head] & c_2 \arrow[ur, no head] \arrow[dr, no head] & & c_2' \arrow[r, no head] & c_3 \arrow[ur, no head] \arrow[dr, no head] & & c_3' \\
 & t_1(\hat{x}_3)\ \ t_1(\bot_3) \arrow[ur, no head] & & & t_2(\hat{x}_3)\ \ t_2(\top_3) \arrow[ur, no head] & & & t_3(\hat{x}_3)\ \ t_3(\bot_3) \arrow[ur, no head] &
\end{tikzcd}$$

$$\bot_1 \;\text{——}\; \hat{x}_1 \;\text{——}\; \top_1$$

$$\bot_2 \;\text{——}\; \hat{x}_2 \;\text{——}\; \top_2$$

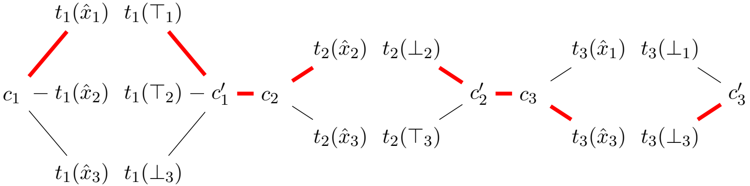
$$\bot_3 \;\text{——}\; \hat{x}_3 \;\text{——}\; \top_3$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$\perp_1$ —— $\hat{x}_1$ —— $\top_1$

$3n - 1$ equations for clauses

$\perp_2$ —— $\hat{x}_2$ —— $\top_2$

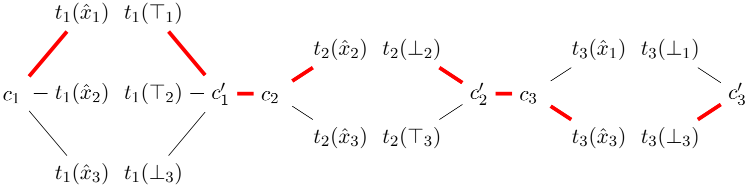
$\perp_3$ —— $\hat{x}_3$ —— $\top_3$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \leadsto a = b \in E)$

Target Equation $c_1 = c_3'$



$\bot_1$ —— $\hat{x}_1$ —— $\top_1$

$\bot_2$ —— $\hat{x}_2$ —— $\top_2$

$\bot_3$ —— $\hat{x}_3$ —— $\top_3$

$3n - 1$ equations for clauses

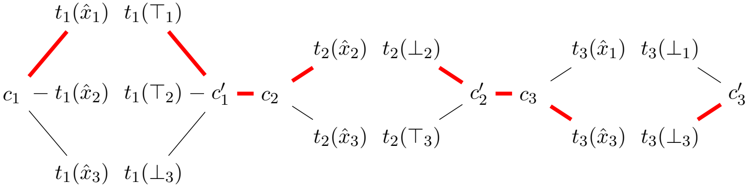
m equations for assignment

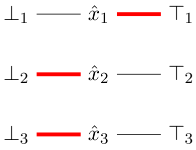
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$





$3n - 1$ equations for clauses

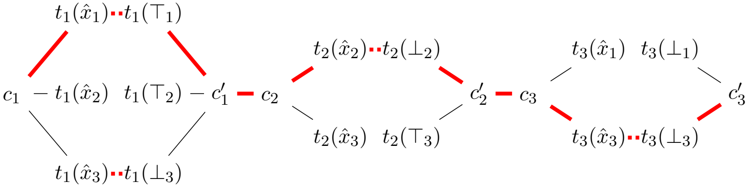
m equations for assignment

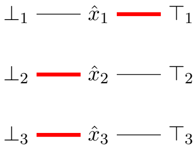
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c'_3$





$3n - 1$ equations for clauses

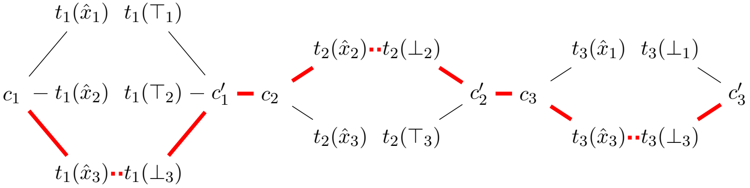
m equations for assignment

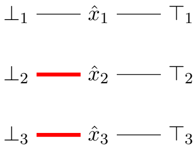
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$





$3n - 1$ equations for clauses

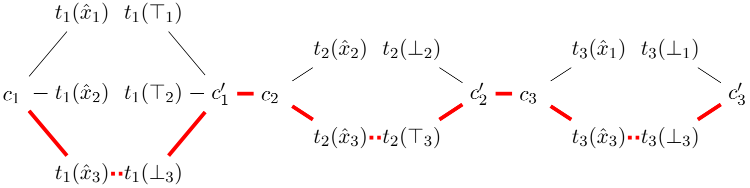
m equations for assignment

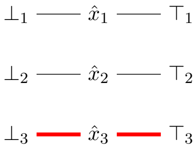
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$





$3n - 1$ equations for clauses

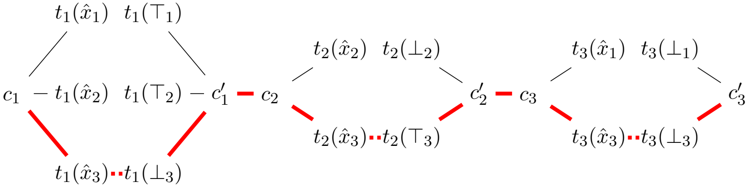
m equations for assignment

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$\bot_1$ —— $\hat{x}_1$ —— $\top_1$

$\bot_2$ —— $\hat{x}_2$ —— $\top_2$

$\bot_3$ —— $\hat{x}_3$ —— $\top_3$

$3n - 1$ equations for clauses

m equations for assignment

Force total assignment by adding clauses

$(x_1 \vee \neg x_1)$,$(x_2 \vee \neg x_2)$,$(x_3 \vee \neg x_3)$

$3m$ equations for extra clauses

Proof Arguments

- Translate assignment $\mathcal{I}$ to subset of equations E':

$$\begin{aligned} x_i \in \mathcal{I} &\Leftrightarrow \hat{x}_i = \top_i \in E' \\ \neg x_i \in \mathcal{I} &\Leftrightarrow \hat{x}_i = \bot_i \in E' \end{aligned}$$

- Every short explanation contains the translation of an assignment

Proof Arguments

- Translate assignment $\mathcal{I}$ to subset of equations E':

$$x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \top_i \in E'$$
$$\neg x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \bot_i \in E'$$

- Every short explanation contains the translation of an assignment

- Satisfying assignments translate to short explanations

Proof Arguments

- Translate assignment $\mathcal{I}$ to subset of equations E':

$$x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \top_i \in E'$$
$$\neg x_i \in \mathcal{I} \Leftrightarrow \hat{x}_i = \bot_i \in E'$$

- Every short explanation contains the translation of an assignment

- Satisfying assignments translate to short explanations

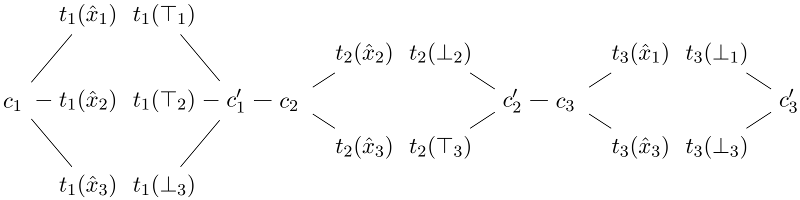
- Non satisfying assignments do not translate to explanations

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$$\bot_1 \text{ --- } \hat{x}_1 \text{ --- } \top_1$$

$$\bot_2 \text{ --- } \hat{x}_2 \text{ --- } \top_2$$

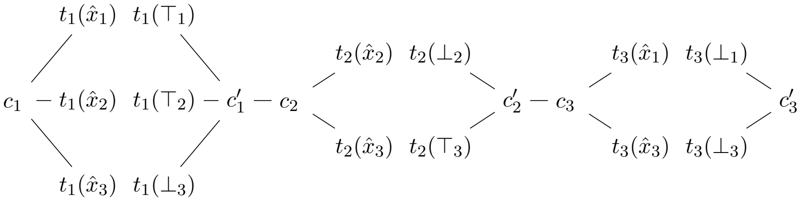
$$\bot_3 \text{ --- } \hat{x}_3 \text{ --- } \top_3$$

Example of Reduction

$$\phi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$$\bot_1 \;\text{———}\; \hat{x}_1 \;\text{———}\; \top_1$$

Assignment:

$$\bot_2 \;\text{———}\; \hat{x}_2 \;\text{———}\; \top_2$$

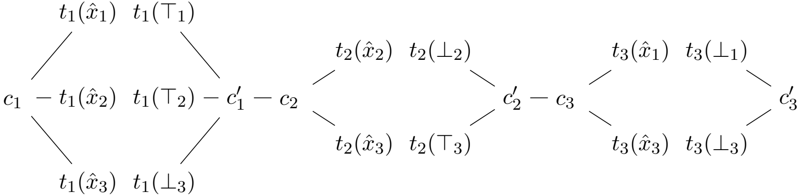
$$\bot_3 \;\text{———}\; \hat{x}_3 \;\text{———}\; \top_3$$

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$



$\bot_1$ —— $\hat{x}_1$ —— $\top_1$

$\bot_2$ —— $\hat{x}_2$ —— $\top_2$

$\bot_3$ —— $\hat{x}_3$ —— $\top_3$

Assignment:

$\mathcal{I}_1 = \{x_1, \neg x_2, \neg x_3\}$

$\mathcal{I}_1 \models \phi$

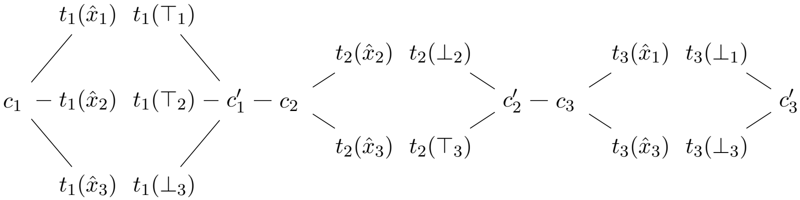
Short explanation E'

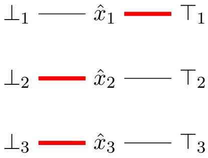
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E ($a - b \rightsquigarrow a = b \in E$)

Target Equation $c_1 = c'_3$





Assignment:

$\mathcal{I}_1 = \{x_1, \neg x_2, \neg x_3\}$

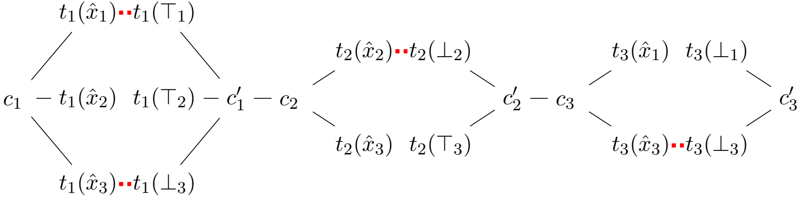
$\mathcal{I}_1 \models \phi$

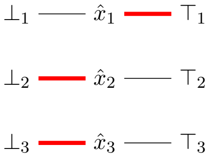
Short explanation E'

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E ($a - b \rightsquigarrow a = b \in E$)
Target Equation $c_1 = c_3'$





Assignment:

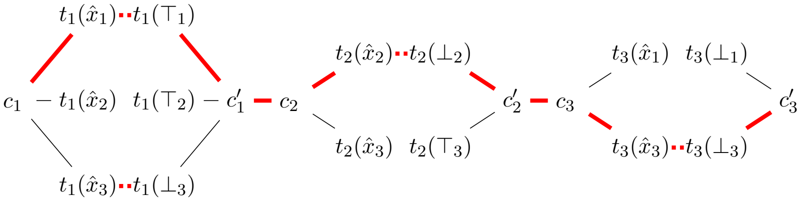
$\mathcal{I}_1 = \{x_1, \neg x_2, \neg x_3\}$
$\mathcal{I}_1 \models \phi$
Short explanation E'

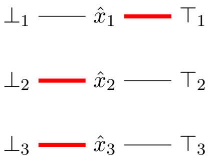
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$





Assignment:

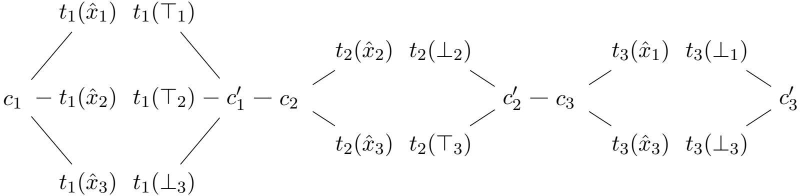
$\mathcal{I}_1 = \{x_1, \neg x_2, \neg x_3\}$
$\mathcal{I}_1 \models \phi$
Short explanation E'

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c'_3$



$$\bot_1 \text{ — } \hat{x}_1 \text{ — } \top_1$$

$$\bot_2 \text{ — } \hat{x}_2 \text{ — } \top_2$$

$$\bot_3 \text{ — } \hat{x}_3 \text{ — } \top_3$$

Assignment:

$\mathcal{I}_2 = \{x_1, \neg x_2, x_3\}$

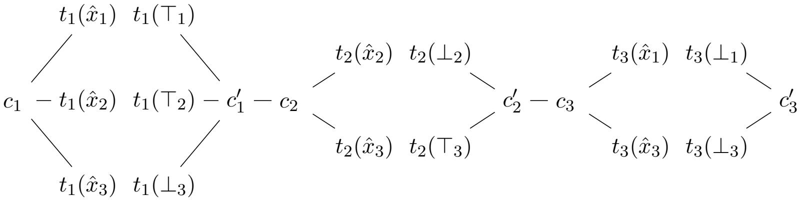
$\mathcal{I}_2 \not\models \phi$

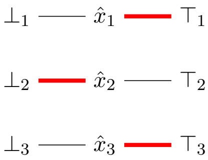
No explanation

Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$
Target Equation $c_1 = c_3'$





Assignment:

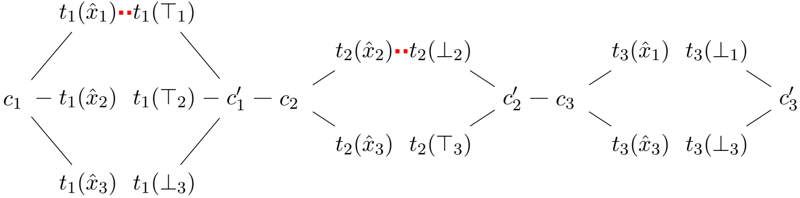
$\mathcal{I}_2 = \{x_1, \neg x_2, x_3\}$
$\mathcal{I}_2 \not\models \phi$
No explanation

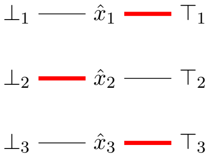
Example of Reduction

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$$

Equations E $(a - b \rightsquigarrow a = b \in E)$

Target Equation $c_1 = c_3'$





Assignment:

$\mathcal{I}_2 = \{x_1, \neg x_2, x_3\}$
$\mathcal{I}_2 \not\models \phi$
No explanation

NP-completeness of short explanation problem

In NP

- Guess explanation and check with congruence closure algorithm

NP-hardness

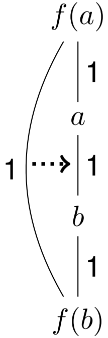
- Reduction of NP-hard problem SAT

ϕ with n clauses and m variables is satisfiable

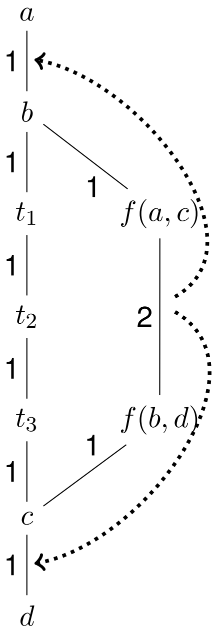
if and only if

There exists an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq 3n+4m-1$

Small explanations as shortest paths



Small explanations as shortest paths



Conclusion

- Small conflict sets are desirable
- Obtaining small conflict sets is NP-complete
- Find algorithms/heuristics to construct small conflict sets

Thank you for your attention !

Questions ?